MATH5111S24: L^p Spaces (Prep for 03/18 & 03/20 Lectures)

Iddo Ben-Ari

1 Intro

Our starting point is a measure space $(\Omega, \mathcal{F}, \mu)$. As usual, we will assume this measure space is complete and to avoid trivialities, we will assume that there exists $A \in \mathcal{F}$ with $\mu(A) \in (0, \infty)$.

We proved that the sum of two measurable functions is measurable and that a constant times a measurable function is measurable. Therefore the set $\mathcal{L}$ of real-valued measurable functions is a vector space with respect to pointwise addition and scalar multiplication. Note that this structure has nothing to do with the measure, just the sigma-algebra.

For $f \in \mathcal{L}$ and $p \in [1, \infty)$. Define

$$\|f\|_p = (\int |f|^p d\mu)^{\frac{1}{p}},$$

and

$$\mathcal{L}^p = \{f \in \mathcal{L} : \|f\|_p < \infty\}.$$

In what follows we will always write q for the conjugate exponent to p, defined through the relation:

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1}$$

For example if $p = 2$, $q = 2$ and if $p = 3$, $q = \frac{3}{2}$, etc.

Note that $q \in (1, \infty]$, and that $q = \infty$ if and only if $p = 1$. The fact that the conjugate of $p = 1$ is $q = \infty$ suggests (as will become apparent through Holder's inequality, Theorem 2.2) that we may want to introduce and define $\mathcal{L}^\infty$. We will do that in Section 4.

The following identity is a restatement of the relation (1):

$$p - 1 = \frac{p}{q}.$$

2 Two Inequalities: Holder & Minkowski

Holder's inequality is among the most important inequalities in analysis. It is a generalization of the Cauchy-schwarz inequality. It is derived from one of the oldest tricks in the playbook, the Arithmetic-Geometric inequality.

Proposition 2.1 (Arithmetic-Geometric Inequality)**.** *Let α, β be nonnegative and $\lambda \in (0,1)$. Then*

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$$

with a strict inequality if and only if $\alpha \neq \beta$.

The proof is equivalent to the strict convexity of the exponential function. Indeed if $\alpha\beta = 0$ the inequality is trivial and otherwise the lefthand side is $e^{\lambda \ln \alpha + (1-\lambda)\ln\beta}$, and the strict convexity of the exponential implies this is $\leq \lambda e^{\ln \alpha} + (1-\lambda)e^{\ln \beta}$, with equality if and only if $\ln \alpha = \ln \beta$.

Theorem 2.2 (Holder's Inequality)**.** *Let $p \in (1,\infty)$ and let q be its conjugate from* (1)*. Let f and g be measurable and nonnegative. Then*

$$\int fg d\mu \leq \|f\|_p \|g\|_q,$$

where the righthand side is defined as 0 if one of the factors is $= 0$. If $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$, then an equality holds if and only $f = 0$ μ-a.e., $g = 0$ μ-a.e. or there exists some $C > 0$ such that $f^p = Cg^q$ μ-a.e.

We note that the condition for equality can be restated as f^p and g^q are linearly dependent, μ-a.e.

Proof. We observe that if $\|f\|_p = 0$ then $f = 0$ μ-a.e., and consequently $fg = 0$ μ-a.e., and therefore both the lefthand side and the righthand side are zero. The same holds if $\|g\|_q = 0$.

We can therefore assume that both $\|f\|_p$ and $\|g\|_q$ are strictly positive. If any of these is infinite, then the inequality is trivial.

We are left with the case $\|f\|_p, \|g\|_q \in (0,\infty)$. We will make yet another reduction which will save some work. Let $F = f/\|f\|_p$ and $G = g/\|g\|_q$. Then

$$\int fg d\mu = \|f\|_p \|g\|_q \int FG d\mu.$$

Therefore Holder's inequality is equivalent to

$$\int FG d\mu \leq 1 \tag{2}$$

We note that the definition of F and G, $\|F\|_p = \|G\|_q = 1$. Let's now use the AGM, Proposition 2.1, with $\lambda = \frac{1}{p}$ and therefore $1 - \lambda = \frac{1}{q}$, $\alpha = F^p(\omega)$ and $\beta = G^p(\omega)$, which then gives

$$(FG)(\omega) \leq \frac{1}{p}F^p(\omega) + \frac{1}{q}G^q(\omega). \tag{3}$$

with a strict inequality on the set $A = \{\omega : F^p(\omega) \neq G^q(\omega)\}$. Now integrate both sides of (3) to obtain (2). An equality holds if and only $\mu(A) = 0$, equivalently

A^c μ-a.e. Now $A^c = \{f^p = \frac{\|f\|_p^p}{\|g\|_q^q} g^q\}$, which implies that for some $C > 0$, $f^p = Cg^q$, μ-a.e. Integrating the last equality gives $\|f\|_p^p = C\|g\|_q^q$, or $C = \|f\|_p^p / \|g\|_q^q$, that is C is determined, and so that if for some $C > 0$, $f^p = Cg^q$ μ-a.e., then A^c μ-a.e., and so an equality holds in Holder's inequality. □

A very important corollary to Holder's inequality is the following, the triangle inequality for $\| \cdot \|_p$:

Theorem 2.3 (Minkowski's Inequality)**.** *Let $p \in (1, \infty)$ and let $f, g \in \mathcal{L}^p$. Then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

Clearly if $f \in \mathcal{L}^p$ and $c \in \mathbb{R}$, then $cf \in \mathcal{L}^p$. Minkowski's inequality implies that if $f, g \in \mathcal{L}^p$, then $f + g \in \mathcal{L}^p$, and so we obtain the following:

Corollary 2.4. *$\mathcal{L}^p$ is a vector space with respect to pointwise addition and scalar multiplication.*

Proof. If the lefthand side is zero there is nothing to prove. We will proceed assuming $\|f + g\|_p > 0$ (apriori it may be infinite). To get the lefthand side we need to integrate the function $|f + g|^p$. Use the triangle inequality to obtain

$$|f + g|^p = |f + g||f + g|^{p-1} \leq (|f| + |g|)|f + g|^{p-1}. \tag{4}$$

Now $|f + g| \leq 2\max(|f|, |g|)$, and so

$$|f + g|^{p-1} \leq 2^{p-1} \max(|f|^{p-1}, |g|^{p-1}).$$

Recalling that $p - 1 = \frac{p}{q}$, it follows that

$$(|f + g|^{p-1})^q \leq 2^p \max(|f|^p, |g|^p) \leq 2^p(|f|^p + |g|^p).$$

Since $f, g \in \mathcal{L}^p$, $|f + g|^{p-1} \in \mathcal{L}^q$. Now,

$$\||f + g|^{p-1}\|_q^q = \int (|f + g|^{p-1})^q = \int |f + g|^p d\mu = \|f + g\|_p^p,$$

that is

$$\||f + g|^{p-1}\|_q = \|f + g\|_p^{p/q}. \tag{5}$$

Now integrate both sides of (4)

$$\begin{aligned} \|f + g\|_p^p &= \int |f + g|^p d\mu \\ &\leq \int |f||f + g|^{p-1} d\mu + \int |g||f + g|^{p-1} d\mu \\ &\leq \|f\|_p \|f + g\|_p^{p/q} + \|g\|_p \|f + g\|_p^{p/q}, \end{aligned}$$

where the last line was obtained from Holder's inequality and (5). Divide by sides by $\|f + g\|_p^{p/q} = \|f + g\|_p^{p-1}$ to obtain the result. □

3 L^p, a Complete Normed Space

3.1 Normed Spaces

Recall the following definition:

Definition 1. *A semi-norm $\|\cdot\|$ on a vector space V is a mapping $\|\cdot\| \to [0,\infty)$ with the following properties*

1. *(homogeneity) $\|cv\| = |c|\|v\|$, for all $c \in \mathbb{R}$, and $v \in V$.*
2. *(triangle inequality) $\|v+u\| \leq \|v\| + \|u\|$ for all $v, u \in V$.*

A semi-norm is a norm if in addition it satisfies

0. *$\|v\| = 0$ if and only if $v = 0$.*

If $\|\cdot\|$ is a norm, then we call $(V, \|\cdot\|)$ a normed space.

Now $\|\cdot\|_p$ is a semi-norm on the vector space $\mathcal{L}^p$. In general, it is not a norm. Consider the Lebesgue measure: $\|\mathbf{1}_{\mathbb{Q}}\|_p = \|0\|_p$, though these two functions are distinct elements of $\mathcal{L}$. We remedy this through a simple construction of a quotient space.

Suppose that V is a vector space and $\|\cdot\| : V \to [0,\infty)$ is a semi-norm on V. Let $V_0 = \{v \in V : \|v\| = 0\}$. Note that V_0 is automatically a subspace of V because of the semi-norm properties.

For every $v \in V$, let $v + V_0 = \{v + v_0 : v_0 \in V_0\}$. Then for $v, v' \in V$ either $v - v' \in V_0$, in which case $v + V_0 = v' + V_0$ or $v - v' \notin V_0$, in which case $(v + V_0) \cap (v' + V_0) = \emptyset$ (check!). Let V/V_0 be the set $\{v + V_0 : v \in V\}$ (in other words: the relation $v - v' \in V_0$ is an equivalence relation on V, and V/V_0 is the set of equivalence classes). We call V/V_0 the quotient of V over V_0.

We can equip the quotient space V/V_0 with a vector space structure in the most obvious way: the addition of $(v + V_0) + (u + V_0)$ is defined as $(u + v) + V_0$ and the scalar multiplication $c(v + V_0)$ is defined as $(cv) + V_0$. The zero element in this vector space, $0_{V/V_0}$, is $0 + V_0$. Abusing notation, extend $\|\cdot\|$ to V/V_0 by letting $\|v + V_0\| = \|v\|$. We will show that this mapping is well defined and is a norm.

First, let's show it is well-defined. Suppose $v + V_0 = v' + V_0$, then $\|v\| = \|v' + \underbrace{(v - v')}_{\in V_0}\| \leq \|v'\| + \|v - v'\| = \|v'\|$, due to the triangle inequality. As the inequality holds with the roles of v and v' interchanged, it follows that $\|v'\| = \|v\|$ and therefore the mapping is well-defined.

It immediately follows from the definition that $\|\cdot\|$ is a semi-norm on V/V_0. To show it is a norm, observe that $\|v + V_0\| = 0$ if and only if $\|v\| = 0$ if and only if $v \in V_0$ if and only if $v + V_0 = 0_{V/V_0}$. Therefore $(V/V_0, \|\cdot\|)$ is a normed space.

Going back to our main topic. The mapping $\|\cdot\|_p$ is a semi-norm on $\mathcal{L}^p$. Indeed, from the definiton, if $f \in \mathcal{L}^p$ and $c \in \mathbb{R}$, then $\|cf\|_p = |c|\|f\|_p$. The triangle inequality for $\|\cdot\|_p$ is Minkowski's inequality, Theorem 2.3. The normed space $(\mathcal{L}^p/\mathcal{L}_0^p, \|\cdot\|_p)$ obtained through the construction above is called L^p. Here $\mathcal{L}_0^p = \{f \in \mathcal{L}^p : \|f\|_p = 0\}$, namely, all functions in $\mathcal{L}^p$ (equivalently, $\mathcal{L}$), which are 0 μ-a.e. An element in L^p is a set of the form $\{f + h : h = 0,\ \mu-\text{ a.e.}\}$, namely all functions equal to f μ-a.e. It would be convenient to denote this element by $[f]$.

We turn to a very important property of our newly minted normed space, completeness.

Definition 2. *Let $(V, \|\cdot\|)$ be a normed space.*

1. *A sequence $(v_n : n \in \mathbb{N})$ in V is convergent if there exists $v \in V$ such that $\lim_{n\to\infty} \|v_n - v\| = 0$, in which case we say that the sequence has a limit v or that the sequence converges to v, denoted by $\lim_{n\to\infty} v_n = v$.*

2. *A sequence $(v_n : n \in \mathbb{N})$ is a Cauchy sequence if for every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that $\|v_n - v_{n'}\| \le \epsilon$ for all $n, n' \ge N$. Equivalently, $\lim_{n\to\infty} \sup_{m\in\mathbb{N}} \|v_{n+m} - v_n\| = 0$.*

3. *$(V, \|\cdot\|)$ is complete if every Cauchy sequence is convergent.*

Note that it is very easy to see that every convergent sequence is Cauchy. Yet not every normed space is complete. For example, $\mathbb{R}^d$ with the Euclidean norm $\|(x_1, \dots, x_d)\| = \sqrt{x_1^2 + \cdots + x_d^2}$ is complete, yet $\mathbb{Q}^d$ with the same norm is clearly not complete. Note that the former is $\mathcal{L}^2$ with μ being the counting measure on $\{1, \dots, d\}$.

We also note that it immediately follows from the definition that every Cauchy sequence is bounded in the following sense. If $(v_n : n \in \mathbb{N})$ is Cauchy, then $\sup_n \|v_n\| < \infty$. Indeed, pick n_1 such that $\sup_m \|v_{n_1+m} - v_{n_1}\} \le 1$. Then for $n \le n_1$, $\|v_n\| \le \max(\|v_1\|, \dots, \|v_{n_1}\|)$ and for $n > n_1$, $\|v_n\| \le \|v_{n_1}\| + 1$.

3.2 Completeness of L^p

In the last section we constructed a normed vector space $(L^p, \|\cdot\|_p)$. We briefly describe its structure. For every $f \in \mathcal{L}^p$, let $[f]$ denote all functions in $\mathcal{L}^p$ (or more generally $\mathcal{L}$) which are equal to f μ-a.e. Each of these sets is an element in L^p. Addition in L^p and scalar multiplication are defined by the rules $[f + g] = [f] + [g]$ and $c[f] = [cf]$. The norm of $[f]$ is $\|f\|_p$. We prove the following:

Theorem 3.1. *$(L^p, \|\cdot\|_p)$ is a complete metric space.*

Proof. We only need to prove completeness.

1. Prep. Let $([f_n] : n \in \mathbb{N})$ be a sequence in L^p. Clearly, there exists $[f] \in L^p$ such that $\lim_{n\to\infty}[f_n] = [f]$ if and only if $\lim_{n\to\infty} \|[f_n] - [f]\|_p = 0$. From the definition of the norm $\|\cdot\|_p$ on L^p, the latter holds if and only if $\lim_{n\to\infty} \|f_n - f\|_p = 0$ (of course we could take instead any $f'_n \in [f_n], f' \in [f]$).

2. Candidate for f. Yes, we find a candidate for f. Take any subsequence $(n_k : k \in \mathbb{N})$ tending to infinity. Then

$$f_{n_k} = f_{n_1} + \sum_{l=1}^{k-1} (f_{n_{l+1}} - f_{n_l}). \tag{6}$$

Therefore,

$$|f_{n_k}| \leq |f_{n_1}| + \sum_{l=1}^{k-1} |f_{n_{l+1}} - f_{n_l}|.$$

It follows from Minkowski's inequality, Theorem 2.3, that

$$\|f_{n_k}\|_p \leq \|f_{n_1}\|_p + \sum_{l=1}^{k-1} \|f_{n_{l+1}} - f_{n_l}\|_p.$$

That's true for any subsequence. We now pick a subsequence so that $(f_{n_l} : l \in \mathbb{N})$ converges μ-a.e. Pick $n_1 = \min\{n : \sup_m \|f_{n+m} - f_n\|_p < 4\}$, and continue inductively, letting $n_{l+1} = \min\{n > n_l : \sup_m \|f_{n+m} - f_n\|_p < 4^{-(l+1)}\}$. This is possible due to the definition of a Cauchy sequence.

Let $A_{l+1} = \{|f_{n_{l+1}} - f_{n_l}| \geq 2^{-(l+1)}) \leq 2^{l+1}\}$. Now by Markov's inequality,

$$\mu(A_{l+1}) \leq 2^{l+1} \|f_{n_{l+1}} - f_{n_l}\|_p \leq 2^{l+1} 4^{-(l+1)} = 2^{-(l+1)}.$$

Therefore, the series $\sum \mu(A_l)$ converges, and in particular $\lim_{n\to\infty} \sum_{l\geq n} \mu(A_l) = 0$.

$$\mu(\limsup A_l) = \mu(\cap_{n=1}^{\infty} \cup_{l\geq n} A_l) \leq \mu(\cup_{l\geq n} A_l) \leq \sum_{l\geq n} \mu(A_l) = 0.$$

In other words, for all but finitely many l's, $|f_{n_{l+1}} - f_{n_l}| < 2^{-(l+1)}$, μ-a.e. In particular, $\sum_{l=1}^{\infty} |f_{n_{l+1}} - f_{n_l}|$ converges μ- a.e. or, the series whose partial sums appear in (6) converges absolutely, μ-a.e. As a result, $\lim_{k\to\infty} f_{n_k}$ converges μ-a.e. Denote its limit by f.

3. Candidate in $\mathcal{L}^p$. This is basically Fatou's lemma which states:

$$\liminf \int |f_{n_k}|^p d\mu \geq \int \liminf |f_{n_k}|^p d\mu = \int |f|^p d\mu.$$

As a result, $\liminf \|f_{n_k}\|_p \geq \|f\|_p$. The lefthand side is finite because our sequence is Cauchy hence bounded.

4. Convergence of subsequence in L^p. Fix some k. Then repeating the argiment from the previous step,

$$\liminf_{l\to\infty} \int |f_{n_l} - f_{n_k}|^p d\mu \geq \int |f_{n_k} - f|^p d\mu.$$

That is $\liminf_{l\to\infty} \|f_{n_k} - f_{n_l}\|_p \geq \|f_{n_k} - f\|_p$. Therefore,

$$\sup_{m\in\mathbb{N}} \|f_{n_k} - f_{n_k+m}\|_p \geq \|f_{n_k} - f\|_p,$$

As $k \to \infty$ the lefthand side tends to 0 as our original sequence is Cauchy, and therefore $\lim_{k\to\infty} \|f_{n_k} - f\|_p = 0$.

5. Convergence of full sequence. We have finally arrived at our last step. Sit back and relax. It's all triangle inequality. For every $n \geq n_1$ there exists a unique k such that $n_k \leq n < n_{k+1}$. Now

$$\|f_n - f\|_p \leq \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p.$$

As $n \to \infty$, $k \to \infty$. Therefore, the first summand on the righthand side tends to 0 because our sequence is Cauchy. The second also tends to 0 because of the previous step. Done. □

4 L^∞

In this section we complete the description of the L^p spaces by introducing the space L^∞. We begin with some motivation. A simple calculus exercise shows that if $a_1, \ldots, a_d$ are real numbers then $\lim_{p\to\infty}(\sum_{n=1}^d |a_n|^p)^{1/p} = \max_{n=1\ldots,d} |a_n|$. If we equip the finite set $\{1, \ldots, d\}$ with the counting norm, then the lefthand side can be viewed as the limit of the L^p-norm of the function $n \to a_n$ as $p \to \infty$. The normed space $(L^\infty, \|\cdot\|_\infty)$ will be a generalization of this maximum.

For $f \in \mathcal{L}$, let

$$\|f\|_\infty = \inf\{L : \mu(|f| > L) = 0\}.$$

Of course, $\|f\|_\infty \leq \sup |f|$. A good example to remember is one we have seen before. Consider a Lebesgue measure. Then $\|\mathbf{1}_\mathbb{Q}\|_\infty = 0 < 1 = \sup |\mathbf{1}_\mathbb{Q}$. As before we define $\mathcal{L}^\infty$ as $\{f \in \mathcal{L} : \|f\|_\infty < \infty\}$.

$\mathcal{L}^\infty$ is a vector space with respect to addition and scalar multiplication of functions and $\|\cdot\|_\infty$ is a semi-norm. The proofs are much simpler than for $\|\cdot\|_p$, where we had to get through Holder's inequality to obtain the triangle inequality, Minkowski's inequality. Let's show the triangle inequality for $\|\cdot\|_\infty$. Let $f, g \in \mathcal{L}^\infty$, and let M_f and M_g be any real numbers strictly larger than $\|f\|_\infty$ and $\|g\|_\infty$, respectively. Then $\mu(|f| > M_f) = 0$ and $\mu(|g| > M_g) = 0$. If $|f + g| > M_f + M_g$, then $|f| + |g| > M_f + M_g$, which implies $|f| > M_f$ or $|g| > M_g$. Therefore the set $\{|f + g| > M_f + M_g\}$ is contained in the set of measure zero $\{f > M_f\} \cup \{g > M_g\}$. This implies

$$\|f + g\|_\infty \leq M_f + M_g.$$

Taking the infimum over allowed values of M_f and M_g and using the definition of $\|\cdot\|_\infty$ then gives

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Functions in $\mathcal{L}^\infty$ are also called essentially bounded (with respect to the given measure): bounded, with the exception of a set of measure zero. Consider again the Lebesgue measure. Let $f(x) = \frac{1}{|x|}$ if x is nonzero rational and 0 otherwise. Then $\|f\|_\infty = 0$, although f is unbounded. It is essentially bounded.

We note that for $f \in \mathcal{L}^\infty$, $|f| \le \|f\|_\infty$ μ-a.e.

Repeating the construction of L^p we obtain the normed space $(L^\infty, \|\cdot\|_\infty)$ where each element in L^∞ is of the form $\{f + h : f \in \mathcal{L}^\infty, h = 0,\ \mu \text{ a.e.}\}$, a set we denote by $[f]$, as usual.

Next we want to prove that $(L^\infty, \|\cdot\|_\infty)$ is complete. This is much easier than for L^p, as convergence in this space is uniform convergence, except on a set of measure zero. Indeed, let $([f_n] : n \in \mathbb{N})$ be a Cauchy sequence. Let $A_n = \{|f| > \|f\|_\infty\}$ and let $A = \cup_{n=1}^\infty A_n$. Then $\mu(A) = 0$ and on A^c, $|f_n| \le \|f_n\|_\infty$. In particular for every $\omega \in A^c$ and every $m, n \in \mathbb{N}$ we have that

$$|f_n - f_m|(\omega) \le \|f_n - f_m\|_\infty.$$

Therefore, for $\omega \in A^c$, the numerical sequence $(f_n(\omega) : n \in \mathbb{N})$ is Cauchy (in $\mathbb{R}$) and therefore converges to some limit $f(\omega)$. Moreover for all $\omega \in A^c$,

$$|f_n - f| = \lim_{m\to\infty} |f_n - f_{n+m}| \le \sup_{m\in\mathbb{N}} \|f_n - f_{n+m}\|_\infty,$$

therefore the convergence is uniform on A^c, and since $\mu(A) = 0$, this implies $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

The last result we would like to prove is Holder's inequality. This is even simpler. Let $f \in \mathcal{L}^1$ and $g \in \mathcal{L}^\infty$. Then

$$|\int fg d\mu| \le \int |f| \|g\|_\infty d\mu = \|f\|_1 \|g\|_\infty.$$