MATH5111S24: Differentiation (completion of 04/24 Lecture

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1 Introduction

The classical fundamental theorem of calculus (FTC) establishes the connection between to basic operations: differentiation and Riemann integration. It has two parts.

- The first, FTC I, states that if $f \in C[a,b]$, the set of continuous functions on $[a,b]$, then $F(x) = \int_a^x f(y)dy$ is differentiable on $[a,b]$ and its derivative is f.
- The second, FTC II, states that if $F \in C^1[a,b]$, the set of functions with derivative in $C[a,b]$, then in fact $F(x) - F(a) = \int_a^x F'(y)dy$.

If D is the differentiation operator and I is integration from a, defined on appropriate domains, then FTC I states that $D \circ I$ is the identity mapping on $C[a,b]$, and FTC II states that $I \circ D$ is the identity on $C^1[a,b]$, up to an addivitive constant (it is the identity on the subspace of $C^1[a,b]$ consisting of functions which are equal to 0 at a).

The FTC is a fundamental tool in analysis as they give us very important results like the substitution formula and integration by parts. Nevertheless, it is restricted to integrals of continuous functions (more more generally piecewise continuous).

In these notes we develop analogous results for Lebesgue integration with respect to the Lebesgue measure, results which we will also loosely refer to as FTC I & II. The results, specifically Theorem 2.4 will provide a complete description of those functions which are integrals of functions in L^1 as the set of absolutely continuous functions.

Before we continue we present the following conventions.

- We write $L^1[a,b]$ for L^1 with respect to the Lebesgue measure restricted to the compact interval $[a,b]$.
- The Lebesgue-Stiltjes measure associated with a nondecreasing and right-continuous function $G : [a,b] \to \mathbb{R}$ is the measure corresponding to the extension of G to $\mathbb{R}$ obtained by making it constant on $(-\infty, a]$ and on $[b, \infty)$. The support of this measure is a subset of $[a,b]$.

2 The Fundamental Theorem of Calculus

In class, we prove the following through Hardy-Littlewood's Maximal inequality:

Theorem 2.1 (FTC I)**.** *Let $f \in L^1[a,b]$. Define $F : [a,b] \to \mathbb{R}$ through*

$$F(x) = \int_{[a,x]} f dm. \tag{1}$$

Then F is differentiable m-a.e and $F' = f$ m-a.e.

Note that the function F defined in (1) is continuous due to the DCT, and therefore continuity is a necessary condition for a function F to be of the form given in (1). As we will show in the next example, continuity is not sufficient.

So what about continuous F? Even differentiability is not enough, as the following example shows.

Example 1.

$$F(x) = \begin{cases} \frac{x}{\ln(1/x)} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Check that the function F is differentiable on $[0,1]$. We show that there is no $f \in L^1[0,1]$ such that (1) holds. We argue by contradiction, assuming (1) holds. For $k = 1, 2, \ldots,$ let $x_k = \frac{1}{2\pi k}$, and let $y_k = \frac{1}{2\pi k + \pi/2}$. Then for all k, $0 < x_{k+1} < y_k < x_k < 1$. For all $k \in \mathbb{N}$, $F(x_k) = 0$ and so

$$|\int_{[y_k, x_k]} f dm| = |F(x_k) - F(y_k)| = |\frac{y_k}{\ln(1/y_k)} \sin(1/y_k)| \geq \frac{1}{(2\pi k + \frac{1}{2}) \ln(2\pi k + \frac{1}{2})}.$$

Therefore

$$\infty > \int_{[0,1]} |f| dm \geq \int_{\cup_{k=1}^{\infty} [y_k, x_k]} |\int f dm| = \sum_{k=1}^{\infty} \frac{1}{(2\pi k + \frac{1}{2}) \ln(2\pi k + \frac{1}{2})} = \infty,$$

a contradiction.

We now present what we will later show is a sufficient condition for F to be of the form (1).

Definition 1. *Let $I \subseteq \mathbb{R}$ be an interval. A function $F : I \to \mathbb{R}$ is absolutely continuous if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for every $N \in \mathbb{N}$ and nonoverlapping intervals $I_1, \ldots, I_N$ with I_j having endpoints $a_j, b_j \in I$ and satisfying $m(\cup_{j=1}^N I_j) \leq \delta$, we have $\sum_{j=1}^N |F(b_j) - F(a_j)| < \epsilon$. The set of absolutely continuous functions on I is denoted by $AC(I)$.*

Example 2.

1. *Recall that F is Lipschitz if there exists some $C \in [0, \infty)$ so that $|F(x) - F(y)| \leq C|x - y|$. Clearly, any Lipschitz function on $[a,b]$ is in $AC[a,b]$, with $\delta = \frac{\epsilon}{C}$ (if $C = 0$ any δ will do). Using the Mean Value Theorem we see that any function which is differentiable with bounded derivative is Lipschitz.*

2. *Another example for absolutely continuous function is the following. Let $f \in L^1[a,b]$, and let $F(x) = \int_{[a,x]} f dm$ be the function defined in* (1). *Your work in Assignment #3 Problem #3 shows (unknowingly for some at the time) that $F \in AC[a,b]$. Indeed, you showed the for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $m(A) < \delta$, then $\int_A |f| dm < \epsilon$. Choosing $A = \cup_{j=1} I_j$, where $I_1, \ldots, I_N$ are nonoverlapping intervals, with I_j having endpoints $a_j, b_j \in [a,b]$, and $m(A) = \sum_{j=1}^N |b_j - a_j| \le \delta(\epsilon)$, then*

$$\sum_{j=1}^N |F(b_j) - F(a_j)| = \sum_{j=1}^N |\int_{[a_j,b_j]} f dm \le \sum_{j=1}^N \int_{[a_j,b_j]} |f| dm = \int_A |f| dm < \epsilon.$$

FTCII, Theorem 2.4 will show the converse: all functions in $AC[a,b]$ are obtained through (1).

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Before we state FTCII we wish to expand a little on properties of absolutely continuous functions. Recall that for a function f and a subset of its domain A, we define $f(A)$, the image of A under f, as $f(A) = \{f(a) : a \in A\}$.

Proposition 2.2. *Let $F \in AC[a,b]$. Then for every $N \subseteq [a,b]$, with satisfying $m(N) = 0$, $m(F(N)) = 0$.*

Corollary 2.3. *Let $F \in AC[a,b]$. Then for every $A \in \mathcal{L}$, $F(A) \in \mathcal{L}$.*

To see why the corollary holds, recall that $A = H \cup N$, where H is F_σ (countable union of closed) and $m(N) = 0$. Thus, $F(A) = F(H) \cup F(N)$. Since F is continuous is maps compact sets into compact sets. As H is a countable union of compact sets (closed subsets of $[a,b]$), $F(H)$, its image under F, is a countable union of compact sets, hence F_σ. From the proposition $F(N) \in \mathcal{L}$. Therefore $F(A) = F(H) \cup F(N) \in \mathcal{L}$.

Proof of Proposition 2.2. Suppose first that $F \in AC[a,b]$, and let $N \in \mathcal{L}$ satisfy $m(N) = 0$. Let $\epsilon > 0$ and let $(I_j : j \in \mathbb{N})$ be a disjoint union of (relatively) open intervals such that $N \subseteq \cup_{j=1}^\infty I_j$ and $m(\cup_{j=1}^\infty I_j) < \delta(\epsilon)$, where δ is as in Definition 1. Let a_j and b_j be the left, respectively right endpoint of I_j. Let x_j, y_j be points in $[a_j, b_j]$ where the minimum and maximum of F are attained, respectively. Then $F(I_j) \subseteq [F(x_j), F(y_j)]$, and so $F(N) \subseteq \cup_{j=1}^\infty [F(x_j), F(y_j)]$. Moreover, the countable union of nonoverlapping intervals with endpoints x_j and y_j, is contained in $\cup_{j=1}^\infty \bar{I}_j$ (where $\bar{I}$ is the closure of I), which differs from $\cup_{j=1}^\infty I_j$ only on a countable set, and therefore has Lebesgue measure equal to $m(\cup_{j=1}^\infty I_j)$ $< \delta(\epsilon)$ and it follows from the definition now that $\sum_{j=1}^\infty |F(y_j) - F(x_j)| < \epsilon$. By since $F \subset \cup_{j=1}^\infty [F(x_j), F(y_j)]$ and $\epsilon > 0$, it follows that the outer Lebesgue measure of F is 0, hence $F \in \mathcal{L}$ and $m(F(N)) = 0$.

□

We are ready to state the main result of this section which also include Theorem 2.1

Theorem 2.4 (FTC I & II)**.** *Let $G : [a,b] \to \mathbb{R}$. Then $G \in AC[a,b]$ if and only if there exists $f \in L^1[a,b]$ such that $G(x) - F(a) = \int_{[a,x]} f dm$. In this case, $G' = f$, m-a.e.*

We will defer the proof of the theorem to Section 5.

3 Differentiation of Measures

Recall that for $x \in \mathbb{R}$ and $r \geq 0$, $B_r(x)$, the ball with center x and radius r is the open interval $(x-r, x+r)$. Clearly $m(B_r(x)) = 2r$.

Theorem 3.1. *Let λ be a finite Borel measure on $\mathbb{R}$. Then*

$$\limsup_{r\to 0+} \frac{\lambda(B_r(x))}{2r} = 0, \ m\text{-a.e.}$$

Corollary 3.2. *Let λ be a finite Borel measure on $\mathbb{R}$, and let $F(x) = \lambda((-\infty, x])$. Then $F' = 0$ m-a.e.*

To see why the Corollary holds, observe that

$$\left|\frac{F(x+\Delta x) - F(x)}{\Delta x}\right| \leq 4\frac{\lambda(B_{2|\Delta x|}(x))}{4|\Delta x|},$$

and from the theorem, as $\Delta x \to 0$ the righthand side tends to 0, m-a.e.

Proof of Theorem 3.1. Let E be such that $\lambda(E) = m(E^c) = 0$, and for every $k \in \mathbb{N}$, let

$$F_k = \{x \in E : \limsup_{r\to 0+} \frac{\lambda(B_r(x))}{2r} > \frac{1}{k}\}.$$

As $\{\limsup_{r\to 0+} \frac{\lambda(B_r(x))}{2r} > 0\} = \cup_{k=1}^{\infty} F_k$, it is enough to show that $m(F_k) = 0$ for all $k \in \mathbb{N}$.

Because λ is also a Lebesgue-Stiltjes measure, it is regular, and in particular for every $\epsilon > 0$, there exists an open U with $E \subseteq U$ such that $\lambda(U) < \epsilon$. For each $x \in F_k$ we pick $r(x) \leq 1$ such that $\frac{\lambda(B_{r(x)}(x))}{2r(x)} > \frac{1}{k}$, and $B_{r(x)}(x) \in U$. Let $(x_j : j \in \mathbb{N})$ be the sequence obtained by the application of Vitali's coverling lemma to $(B_{r(x)}(x) : x \in F_k)$. That is $(B_{r(x_j)}(x_j) : j \in \mathbb{N})$ are disjoint and

$$m(F_k) \leq m(\cup_{x\in F_k} B_{r(x)}(x)) \leq 5m(\cup_{j=1}^{\infty} B_{r(x_j)}(x_j).$$

However, by choice of $r(x)$, and the fact that the intervals in the sequence are disjoint, the righthand side is bounded above by

$$5k\lambda(\cup_{j=1}^{\infty} B_{r(x_j)}(x_j)) \leq 5k\lambda(U) \leq 5k\epsilon.$$

Therefore, $m(F_k) \leq 5k\epsilon$, and as ϵ is arbitrary the result follows. ☐

As an application we have the following:

Theorem 3.3. *Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and let $G(x) = F(x+) = \lim_{y \downarrow x} F(y)$. Then*

1. *G is right continuous.*

2. *G and F are differentiable m-a.e.*

3. *$G' = F'$ m-a.e.*

Proof. Let $x \in \mathbb{R}$ and let $(x_n : n \in \mathbb{N})$ be a sequence of numbers strictly larger than x, decreasing to x. For each n, $G(x) \leq G(x_{n+1}) \leq F(x_n)$. Moreover, all are non-increasing in n and are bounded below by $F(x)$, and therefore have a limit in $\mathbb{R}$. therefore

$$G(x) \leq \lim_{n\to\infty} G(x_{n+1}) \leq \lim_{n\to\infty} F(x_n) = G(x).$$

Therefore G is right-continuous.

Fixing any $M \in \mathbb{N}$, we can replace F with the bounded right continuous function F_M which is equal to F on $[-M, M]$ and is constant on each of the intervals $[M, \infty)$ and $(-\infty, -M]$. Proving the last two statements for F_M for every M clearly implies the corresponding statements for F. Therefore without loss of generality, for the remainder of the proof we assume that F, hence G, is bounded.

Since G is non-decreasing, right-continuous and bounded, there exists a finite LS measure corresponding to G, μ_G. Specifically, $\mu_G((-\infty, x]) = G(x) - G(-\infty)$.

Consider the Lebesgue-Radon-Nikodym decomposition of μ_G with respect to m. It gives us a unique $f \in L^1(m)$, measure λ singular with respect to m such that for every Borel set A

$$\mu_G(A) = \int_A f dm + \lambda(A).$$

Let $G_1(x) = \int_{(-\infty,x]} f dm$ and let $G_2(x) = \lambda((-\infty, x])$. Then $G(x) = G_1(x) + G_2(x)$ By FTC I, $G_1'(x) = f(x)$ m-a.e. and by Corollary 3.2, $G_2'(x) = 0$ m-a.e. and therefore $G'(x) = f(x) + 0$, m-a.e.

To wrap everything up, we will examine the connection between F and G. Let D be the set of discontinuities of F. As F is nondecreasing, D is countable. The set of points where $F(x) \neq G(x) = F(x+)$ is a subset of D. Note that by definition and monotonicity, for $y < x$, $G(y) \leq F(x)$ and therefore $G(x-) \leq F(x)$. Thus, for any $x \in D$ we have $\mu_G(\{x\}) = 0 + \lambda(\{x\}) = G(x) - G(x-) \geq G(x) - F(x)$. Let $x, \Delta x \in \mathbb{R}$. Then

$$\left| \frac{F(x + \Delta x) - F(x)}{\Delta x} - \frac{G(x + \Delta x) - G(x)}{\Delta x} \right| \leq \left| \frac{\lambda(\{x, x + \Delta x\})}{\Delta x} \right| \leq 4 \frac{\lambda(B_{2|\Delta x|}(x))}{4|\Delta x|}.$$

By Theorem 3.1, the righthand side tends to zero m-a.e. This and the differentiability of G m-a.e. imply that F' is differentiable m-a.e. and $F' = G'$ m-a.e. $\square$

We close this section with the a decomposition result. We need to introduce some additional notation. A Borel measure λ is

- Purely atomic if there exists some countable set D such that $\lambda(D^c) = 0$.
- Atom-free if for every $x \in \mathbb{R}$, $\lambda(\{x\}) = 0$.

Any purely atomic measure is a countable (including finite) sum of the form $\sum c_j \delta_{x_j}$, where $c_j \geq 0$ and $x_j \in \mathbb{R}$, and δ_x is the Dirac delta measure $\delta_x(A) = \mathbf{1}_A(x)$. As simple examples of atom-free measures, consider the Lebesgue measure and the LS measure associated with the Cantor function. The support of the latter is the Cantor set which has Lebesgue measure zero, and therefore the two measures are singular.

The main result of this section is the following decomposition theorem.

Theorem 3.4. *Let $G : \mathbb{R} \to \mathbb{R}$ be non-decreasing, right continuous and bounded. Let $D = \{x \in \mathbb{R} : G(x) > G(x-)\}$, the (countable) set of discontinuity points of G and let μ_G be the LS measure corresponding to G. Then*

1. *G' exists m-a.e. and is in $L^1(m)$.*
2. *Let λ_d be the purely atomic measure $\lambda_d = \sum_{y \in D}(G(y) - G(y-))\delta_y$. Then there exists an atom-free measure λ_c, singular with respect to m such that for every Borel set A*

$$\mu_G(A) = \int_A G' dm + \lambda_c(A) + \lambda_d(A).$$

The theorem provides a decomposition of finite Borel measures on $\mathbb{R}$ as unique sum of three finite measures:

1. A measure λ_{ac} which is absolutely continuous with respect to m, the LS measure corresponding to AC function $G_{ac}(x) = \int_{(-\infty,x]} G' dm$.
2. A purely atomic measure λ_d, the LS measure corresponding to the function $G_d(x) = \sum_{y \in (-\infty,x] \cap D}(G(y) - G(y-))$.
3. A measure λ_c, which is both atom-free and singular with respect to m and which is the LS measure corresponding to the continuous nondecreasing function $G_c = G - G_{ac} - G_d$ with the property. Moreover, $G'_c = 0$ m-a.e.

Proposition 3.5. *Let $G : [a, b] \to \mathbb{R}$ be nondecreasing and right-continuous. Then $G \in AC[a, b]$ if and only if the corresponding LS measure μ_G is absolutely continuous with respect to m. In this case, $\frac{d\mu_G}{dm} = G'$, m-a.e.*

Proof. Suppose $\mu_G \ll m$. By construction: the measures λ_c and λ_d in Theorem 3.1 are by definition absolutely continuous with respect to μ_G, hence also with respect to m, and singular with respect to m. Therefore $\lambda_d = \lambda_c = 0$. In particular $G(x) - G(a) = \mu_G((a, x]) = \int_{(a,x]} G' dm$, and Example 2-2 shows that $G \in AC[a, b]$. Conversely, let $G \in AC[a, b]$. Then since G is continuous, for

$a \leq \alpha \leq \beta \leq b$, we have $\mu_G((\alpha, \beta))) = G(\beta) - G(\alpha)$. Now fix $\epsilon > 0$ and let δ be as in Definition 1. Therefore for any relatively open $U \subseteq [a, b]$, the disjoint union of intervals with endpoints $a_j < b_j$, we have $\mu_G(U) \subseteq \cup_{j=1}^{\infty} G(b_j) - G(a_j)$. In particular if $\epsilon > 0$ $\delta(\epsilon)$ is as in Definition 1, and $m(U) < \delta$, then $\mu_G(U) \leq \epsilon$. Let N satisfy $m(N) = 0$. There exists a decreasing sequence of relatively open sets $(U_n : n \in \mathbb{N})$ such that $N \subseteq \cap_{j=1}^{\infty} U_n$, and $m(U_n) \leq \delta(\frac{1}{n})$. As a result of continuity from above of finite measures, we have $\mu_G(N) \leq \lim_{n\to\infty} \mu_G(U_n) \leq \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore $\mu_G \ll m$. □

Proof of Theorem 3.1. As shown in the proof of Theorem 3.3, there exists some measure λ, singular with respect to m so that for every Borel set A

$$\mu_G(A) = \int_A G' dm + \lambda(A).$$

For every $x \in \mathbb{R}$, $\lambda(\{x\}) = \mu_G(x) = G(x) - G(x-)$. In particular and since D is countable, for any Borel set A,

$$\lambda(A \cap D) = \sum_{x \in A \cap D} \lambda(\{x\}) = \lambda_d(A).$$

Let $\lambda_c = \lambda - \lambda_d$. That is $\lambda_c(A) = \lambda(A \cap D^c)$. Then by construction $\lambda_c \perp m$. Finally, if $x \in D$, then $\lambda_c(\{x\}) = \lambda(\emptyset) = 0$ and if $x \in D^c$, then $\lambda_c(\{x\}) = \lambda(\{x\}) = G(x) - G(x-) = 0$. □

4 Functions of Bounded Variation

Definition 2. *A function $F : [a, b] \to \mathbb{R}$ is called of bounded variation if there exists $L \in [0, \infty)$ such that for every $N \in \mathbb{N}$ and partition $a = x_0 < x_1 < \cdots < x_N = b$,*

$$\sum_{k=0}^{N-1} |F(x_{k+1}) - F(x_k)| \leq L. \tag{2}$$

The set of functions on $[a, b]$ which are of bounded variation is denoted by $BV[a, b]$. The supremum of the lefthandside of (2) over all partitions is called the total variation of F and is denoted by $T_F(b)$.

Any nondecreasing function on $[a, b]$ is of bounded variation, because all the differences $F(x_{k+1}) - F(x_k)$ are nonnrgative and therefore (2) is a telescopic sum equal to $F(b) - F(a)$.

To continue our developments, we need some additional notation. Let $F \in BV[a, b]$, and for every $x \in [a, b]$, let $T_F(x) = T_{F|_{[a,x]}}$. That is $T_F(x)$ is the total variation of the function F, restricted to $[a, x]$. Clearly, $0 = T_F(a) \leq T_F(x) \leq T_F(b)$.

We have the following fundamental yet simple result.

Proposition 4.1. *Let $F : [a, b] \to \mathbb{R}$. Then*

1. *The two functions $T_F(x) \pm F(x)$ are non-decreasing, and therefore F is a difference of two non-decreasing functions.*

2. *If F is a difference of two non-decreasing functions, then $F \in BV[a,b]$.*

Proof. Let $\Delta x > 0$. Then

$$T_F(x+\Delta x) \geq T_F(x) + |F(x+\Delta x) - F(x)| \geq T_F(x) \pm (F(x+\Delta x) - F(x)).$$

This gives the first statement.

As for the second statement, if $F = F_1 - F_2$, where both F_1, F_2 are non-decreasing, then the triangle inequality gives that $T_F(b) \leq T_{F_1}(x) + T_{F_2}(x) \leq F_1(b) - F_1(a) + F_2(b) - F_1(a) < \infty$. and the result follows. □

Proposition 4.2. *Let $F \in BV[a,b]$ and let $G(x) = G(x+) = \lim_{y\downarrow x} F(y)$. Then for all $x \in [a,b]$, $T_G(x) \leq T_F(x)$. In particular, $G \in BV[a,b]$.*

Proof. Fix any x, and let $a = x_0 < x_1 < \cdots < x_N = x$. Theorem 3.3 gives that G is right continuous. Fix $\epsilon > 0$. As F by Proposition 4.1 F is a difference of two nondecreasing functions , it has at mostly countably many discontinuities. If a partition point $x_k < b$ discontinuity point of F, we replace it by a continuity point of F x'_k satisfying $x'_k \in (x_k, x_{k+1})$ and $|G(x'_k) - G(x_k)| < \epsilon/2^k$. Otherwise we set $x'_k = x_k$. We then have

$$T_F(x) \geq \sum_{k=0}^{N-1} |F(x'_{k+1}) - F(x'_k)| = \sum_{k=0}^{N-1} |G(x'_{k+1}) - G(x'_k)| \geq \sum_{k=1}^{N-1} |G(x_{k+1}) - G(x_k)| - 2\epsilon.$$

By taking the supremum over all partitions, we have $T_F(x) \geq T_G(x) - \epsilon$, and since ϵ is arbitrary, the result follows. □

Proposition 4.3. *1. $AC[a,b] \subset BV[a,b]$.*

2. $G \in AC[a,b]$ if and only if $T_G(\cdot) \in AC[a,b]$.

Proof. For the first part, let $\epsilon = 1$ and let δ be as in the Definition 1. Note that it follows from the triangle inequality that the lefthand side of (2) does not decrease if we add points to the partition. Let $K = \max\{k \in \mathbb{Z}_+ : a + k\delta/2 < b\}$. Clearly, $K \leq (b-a)/(2\delta)$, and add partition points at $a + k\delta/2,\ k = 0, \ldots, K$. These partition points split $[a,b]$ into at most $K+1$ subintervals, each of which is of length bounded above by $\delta/2$. Now use the definition of AC to conclude that under the refined partition, the lefthand side of (2) is bounded above by $(K+1) * 1$. As K is independent of the choice of the partition, the result follows.

As the incremenets of T_G are by definition larger or equal to the increments of G, if $T_G \in AC[a,b]$, then $G \in AC[a,b]$. For the converse, suppose $G \in AC[a,b]$, fix $\epsilon/2$, and let $\delta = \delta(\epsilon/2)$ be as in the definition (1) for G. Let $I_1, \ldots, I_N$ be disjoint intervals in $[a,b]$ with $\sum_{j=1}^N m(I_j) < \delta$, and let $a_j < b_j$ be the endpoints of I_j. For each j, pick a partition $(x_{j,k} : k = 0, \ldots, N_j)$ $a_j = x_{j,0} <$

$x_{j,1} < \cdots < x_{j,N_j} = b_j$ with the property $\sum_{k=1}^{N_j-1} |G(x_{j,k+1}) - G(x_{j,k})| \geq T_G(b_j) - T_G(a_j) - \epsilon/2^{j+1}$. Therefore,

$$\sum_{j=1}^{N} |T_G(b_j) - T_G(a_j)| \leq \epsilon/2 + \sum_{j=1}^{N} \sum_{k=1}^{N_j-1} |G(x_{j,k+1}) - G(x_{j,k})|.$$

The sum on the righthand side is the sum of differences of G over a finite number of disjoint intervals whose union has measure $< \delta$, and is therefore $\leq \epsilon/2$, and we therefore showed that the lefthand side is bounded above by ϵ, completing the proof. □

Next, let $G \in BV[a,b]$ be right continuous and bounded (note that we do not require G to be nonndecreasing) and let $\mu_{G\pm}$ be the LS measures corresponding to the right-continuous and non-decreasing functions $\frac{T_G \pm G}{2}$, respectively. Then clearly $\mu_G = \mu_{G_+} - \mu_{G_-}$ is a finite signed measure. We have the following.

Theorem 4.4. *Let $G \in BV[a,b]$ be right-continuous. Then*

1. *For all $x \in [a,b]$, $T_G(b) \geq \int |G'|dm$. An equality holds if and only if $G \in AC[a,b]$.*
2. *$\mu_{G_+} \perp \mu_{G_-}$.*

Proof. As $T_G(x + \Delta x) - T_G(x)| \geq |G(x + \Delta x) - G(x)|$, and since both T_G and G are differentiable m-a.e. it immediately follows that $T_G'(x) \geq |G'(x)|$ m-a.e. If μ_{T_G} is the LS measure corresponding to T_G, then

$$T_G(x) = \mu_{T_G}((a,x]) \geq \int_{[a,x]} T_G' dm \geq \int_{[a,x]} |G'|dm. \tag{3}$$

This proves the inequality. We turn to the characterization of the equality. Assume first that $T_G(b) = \int_{[a,b]} |G'|dm$. Then necessarily $T_G(x) = \int_{[a,x]} |G'|dm = \int_{[a,x]} T_G' dm$ for all $x \in [a,b]$. This implies $T_G' = |G'|$ m-a.e. and $T_G' \in L^1[a,b]$. Theorem 2.4, gives that $T_G \in AC[a,b]$. Proposition 4.3 then implies $G \in AC[a,b]$.

Conversely, suppose that $G \in AC[a,b]$, which by Theorem 2.4 implies that for every partition $a = x_0 < x_1 < \cdots < x_N = b$, and every $k = 0, \ldots, N-1$, $|G(x_{k+1}) - G(x_k)| = |\int_{[x_k,x_{k+1}]} G'dm|$. Now let $h : [a,b] \to \{-1,1\}$ be the function equal to 1 on (x_k, x_{k+1}) if $\int_{[x_k,x_{k+1}]} G'dm \geq 0$ and to -1 otherwise. Set the function to 1 on all remaining points. Then h is a step function (piecewise constant / constant on intervals), and moreover,

$$\sum_{k=0}^{N-1} |G(x_{k+1}) - G(x_k)| = \sum_{k=0}^{N-1} \int_{[x_k,x_{k+1}]} G'hdm = \int G'hdm.$$

Letting $\mathcal{U}$ be the set of step functions on $[a,b]$ taking values in $\{-1,1\}$, then we conclude that

$$T_G(b) = \sup_{h \in \mathcal{U}} \int G'hdm \leq \int |G'|dm,$$

where the inequality is due to Holder's inequality for $G' \in L^1[a,b]$ and $h \in L^\infty[a,b]$. Therefore for $x = b$ (equivalently for all $x \in [a,b]$) we have an equality in (3), and the result follows.

We turn to the second part of the statement. The argument is very similar to the one presented in the last paragraph. First, observe that the fact that the incremenets of G are dominated by those of T_G implies $\mu_{G\pm} \ll \mu_{T_G}$ and let $f_\pm = \frac{d\mu_{G\pm}}{d\mu_{T_G}}$. In addition, $f_+ + f_- = 1$, μ_{T_G}-a.e. Then $G(x) - G(a) = \int_{(a,x]}(f_+ - f_-)d\mu_{T_G}$, and in particular for any $a \le x \le y \le b$, $|G(y) - G(x)| = |\int_{(x,y]} f_+ - f_- d\mu_{T_G}|$. Repeat the argument from the above paragraph with m replaced by μ_{T_G} and G' replaced by $f_+ - f_-$, to conclude that

$$T_G(b) = \sup_{h\in\mathcal{U}} \int (f_+ - f_-)hd\mu_{T_G} \le \int |f_+ - f_-|d\mu_{T_G} \le \int 1d\mu_{T_G} = T_G(b).$$

Therefore we have an equality which in turn implies $|f_+ - f_-| = f_+ + f_-(= 1)$ μ_{T_G}-a.e. Equivalently, $f_+f_- = 0$, μ_{T_G}-a.e. Therefore $\mu_{G_-}(\{f_+ = 0\}^c) = \mu_{G_-}(\{f_- = 0\}) = 0$. As by construction $\mu_{G_+}(\{f_+ = 0\}) = 0$, we have $\mu_{G_+} \perp \mu_{G_-}$.

□

5 Proof of Theorem 2.4

Proof. We begin by assuming that $G \in AC[a,b]$. By Proposition 4.3 $G \in BV[a,b]$ and $T_G(\cdot) \in AC[a,b]$. As both functions $G_\pm = \frac{T_G \pm G}{2} \in AC[a,b]$, and are non-decreasing, Proposition 3.5 gives that the respective LS measures $\mu_{G_\pm}$ are both absolutely continuous with respect to m with $f_\pm = \frac{d\mu_{G_\pm}}{dm} = \frac{T_G' \pm G'}{2}$. As

$$G(x) - G(a) = \mu_{G_+}((a,x]) - \mu_{G_-}((a,x]) = \int_{(a,x]} f_+ - f_- dm = \int_{[a,x]} G' dm.$$

The converse was proved as Example 2-2. □