Random Walk with Catastrophes

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[1] based on joint work with R. Schinazi and A. Roitershtein

Outline

1. Introduction.
2. Convergence to stationarity.
3. Upper and lower bounds on convergence.
4. Poisson limit.
5. Cutoff.

Introduction

Why?

- Simplest model involving linear random growth and subcritical branching.
- Interesting behavior initially observed in through simulations (**all** credit to Rinaldo).

Process

$\mathbf{X} = (X_n : n \in \mathbb{Z}_+)$, a MC with state space $\mathbb{Z}_+ = \{0, 1, 2 \dots\}$, representing size of a population evolving in time.

Starting from population of size i

- w.p. p, population increases by 1; and
- w.p. $1-p$, **a binomial catastrophe**: each member of population dies with probability c independently of everything, that is transition to $\text{Bin}(i, 1-c)$.

$$\text{Bin}(i,1-c) \xleftarrow{\;1-p\;} i \xrightarrow{\;p\;} i+1$$

Formula?

$$p(i,j) = \begin{cases} p & j = i+1 \\ (1-p)\binom{i}{j}(1-c)^j c^{i-j} & j \in \{0,\dots,i\} \end{cases}$$

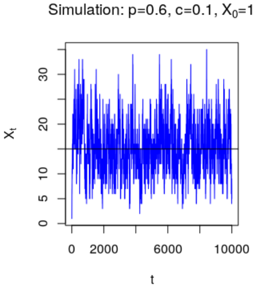
First Calculation

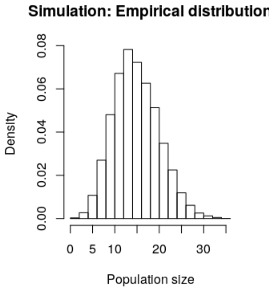
$$\begin{aligned} E_i[X_{t+1}|X_0,\ldots,X_t] &= p(X_t+1)+(1-p)(1-c)X_t \\ &= X_t + p - (1-p)cX_t \\ &= X_t + p(1-\frac{(1-p)c}{p}X_t) \\ \cdots \Rightarrow \lim_{t\to\infty} E_i[X_t] &= \frac{p}{(1-p)c}. \end{aligned}$$

In particular:

- The distributions of $\{X_t : t \in \mathbb{Z}_+\}$ are tight, and so
- The process is positive recurrent and "mean reverting" around $\mu = \frac{p}{(1-p)c}$.

Simulation





Note:

- The process seems to be nearly stationary oscillating around $\mu = \frac{p}{c(1-p)} = 15$, black line.
- The process does not hit 0 at all.

Why?

- The stationary distribution assigns a probability lower than $3 * 10^{-5}$ to 0.
- Process converges to its stationarity distribution very fast. In less than 300 steps it is closer to π than that.
- Bottom line: the $O(1)$ probability of hitting 0 from "low" populations quickly changes to $o(1)$ from "typical" populations.

The Stationary distribution

Shifted Geometric

We say that $G \sim \text{Geom}^-(\rho)$ if

$$P(G = g) = (1-\rho)^g \rho, \;\; g = 0, 1, 2, \ldots.$$

Observation: $G \sim \text{Geom}^-(\rho)$ and $I \sim \text{Ber}(1-\rho)$ independent. Then $I(G+1) \sim G$.

Idea

- Suppose the number of individuals not experiencing a catastrophe yet is G_0.
- After one step this number will be $I(G_0 + 1)$, where is an independent $I \sim \text{Bern}(p)$.
- Due to observation: stationary if $G \sim \text{Geom}^-(1-p)$.

Summary

Let $G_0, G_1, \ldots$ be IID $\sim \text{Geom}^-(1-p)$. The stationary distribution π is the independent sum of

- G_0 individuals who have not experienced a single catastrophe.
- $\text{Bin}(G_1, 1-c)$ - survived exactly one catastrophe
- $\text{Bin}(G_2, (1-c)^2)$ - survived exactly two catastrophes.
- $\ldots$.

Convergence

Total Variation

- The total variation metric between probability measures Q_1, Q_2 on $\mathbb{Z}_+$ is defined as
$$\|Q_1 - Q_2\|_{TV} = \max_{A \subset \mathbb{Z}_+} Q_1(A) - Q_2(A) = \frac{1}{2} \sum_{x \in \mathbb{Z}_+} |Q_1(x) - Q_2(x)|.$$
- Write:
$$d_t(\mu, \mu') = \|P_\mu(X_t \in \cdot) - P_{\mu'}(X_t \in \cdot)\|_{TV}.$$

Coupling

- A process $(\mathbf{X}, \mathbf{X}')$ consisting of two copies of the RW with initial distributions μ, μ', resp.
- The coupling time, $\tau_{coup} = \inf\{t : X_t = X'_t\}$.
- Write $P_{\mu,\mu'}$ for the law of $(\mathbf{X}, \mathbf{X}')$.

Aldous Inequality

$$d_t(\mu, \mu') \leq P_{\mu,\mu'}(\tau_{coup} > t).$$

Our Coupling

The construction

- We assume $\mu = \delta_x, \mu' = \delta_{x'}$ with $0 \leq x \leq x'$.
- Simplest possible:
  - Up: together.
  - Catastrophe: all individuals survive independently.
- Transitions

$$\mathrm{Bin}(i, 1-c) + (0, \mathrm{Bin}(i'-i, 1-c)) \xleftarrow{1-p} (i, i') \xrightarrow{p} (i+1, i'+1)$$

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Summary

- The difference $\Delta_t = X'_t - X_t$ is non-increasing and can only change after a catastrophe, each surviving with probability $1-c$, independently of others.

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- $P_{x,x'}(\Delta_t \in \cdot \,|N_t) \sim \mathrm{Bin}(x'-x, (1-c)^{N_t})$.

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- $P_{x,x'}(\Delta_t \in \cdot \,|N_t) \sim \mathrm{Bin}(x'-x, (1-c)^{N_t})$.
- $\{\tau_{coup} > t\} = \{\Delta_t > 0\} = \{\mathrm{Bin}(x'-x, (1-c)^{N_t}) > 0\}$.

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- The difference $\Delta_t = X'_t - X_t$ is non-increasing and can only change after a catastrophe, each surviving with probability $1-c$, independently of others.
- The number of catastrophes up to time t, $N_t \sim \mathrm{Bin}(t, 1-p)$.
- $P_{x,x'}(\Delta_t \in \cdot | N_t) \sim \mathrm{Bin}(x'-x, (1-c)^{N_t})$.
- $\{\tau_{coup} > t\} = \{\Delta_t > 0\} = \{\mathrm{Bin}(x'-x, (1-c)^{N_t}) > 0\}$.
- $\Rightarrow P_{x,x'}(\tau_{coup} > t) = 1 - E[(1-(1-c)^{N_t})^{x'-x}]$

Upper bound through coupling

Recall,

$$d_t(x,x') \le P_{x,x'}(\tau_{coup} > t) = 1 - E[(1-(1-c)^{N_t})^{x'-x}].$$

Let

$$\alpha = 1 - c(1-p).$$

Upper bound

With some calculus,

Proposition 1

Suppose $0 \le x \le x'$. *Then*

$$d_t(x,x') \le (x'-x)\alpha^t.$$

and

Corollary 1

1. $d_t(x,\pi) \le \left(x - \mu + 2\sum_{y>x}(y-x)\pi(y)\right)\alpha^t$; *and*
2. $d_t(0,\pi) \le \mu\alpha^t$

Lower Bound

Notation

- Recall $\alpha = 1 - c(1-p)$
- Let $\tilde{p} = \dfrac{p}{\alpha} = \dfrac{p}{1-c(1-p)}$.
- Write $P^{\tilde{p}}_{\cdot}, \pi^{\tilde{p}}$, for the respective quantities with parameters $(\tilde{p}, c)$ instead of (p, c).

The bound

- From Proposition 1, $d_t(x, x') \leq (x' - x)\alpha^t$.

Theorem 1

Let $0 \leq x \leq x'$. *Then*

$$d_t(x, x') \geq \alpha^t \max_{j \in \mathbb{Z}_+} \sum_{k=x}^{x'-1} P^{\tilde{p}}_k(X_t = j).$$

Upper and lower bounds give

Corollary 2

$$\max_j \pi^{\tilde{p}}(j) \leq \liminf_{t\to\infty} \frac{d_t(x, x')}{(x'-x)\alpha^t} \leq \limsup_{t\to\infty} \frac{d_t(x, x')}{(x'-x)\alpha^t} \leq 1.$$

Lower bound - Strategy

Goal

$$d_t(x, x') \geq \alpha^t \max_{j \in \mathbb{Z}_+} \sum_{k=x}^{x'-1} P_k^{\tilde{p}}(X_t = j). \tag{1}$$

Stages

Here's our plan

I. Getting the sum.

II. Getting the change of parameter.

Lower bound - I. Sum

Write $I_j = \{0, \ldots, j\}$, $j \in \mathbb{Z}_+$. Then

$$d_t(x, x') \geq P_x(X_t \in I_j) - P_{x'}(X_t \in I_j)$$

Explanation

From definition of total variation, $d_t(x, x') = \max_{A \subset \mathbb{Z}_+} P_x(X_t \in A) - P_{x'}(X_t \in A)$

Lower bound - I. Sum

Write $I_j = \{0, \ldots, j\}$, $j \in \mathbb{Z}_+$. Then

$$d_t(x, x') \geq P_x(X_t \in I_j) - P_{x'}(X_t \in I_j)$$
$$= \sum_{k=x}^{x'-1} P_k(X_t \in I_j) - P_{k+1}(X_t \in I_j)$$

Explanation

Telescoping over all k from x to x'

Lower bound - I. Sum

Write $I_j = \{0, \dots, j\}$, $j \in \mathbb{Z}_+$. Then

$$d_t(x, x') \geq P_x(X_t \in I_j) - P_{x'}(X_t \in I_j)$$

$$= \sum_{k=x}^{x'-1} \underbrace{P_k(X_t \in I_j)}_{(*)} - \underbrace{P_{k+1}(X_t \in I_j)}_{(**)}$$

$$= \sum_{k=x}^{x'-1} E_{k,k+1}[\underbrace{\mathbf{1}_{I_j}(X_t)}_{(*)} - \underbrace{\mathbf{1}_{I_j}(X'_t)}_{(**)}]$$

Explanation

Expressing in terms of our coupling

Lower bound - I. Sum

Write $I_j = \{0, \ldots, j\}$, $j \in \mathbb{Z}_+$. Then

$$\begin{aligned} d_t(x, x') &\geq P_x(X_t \in I_j) - P_{x'}(X_t \in I_j) \\ &= \sum_{k=x}^{x'-1} P_k(X_t \in I_j) - P_{k+1}(X_t \in I_j) \\ &= \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X'_t)] \\ &= \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X'_t), \Delta_t = 1] \end{aligned}$$

Explanation

$\Delta_t \in \{0, 1\}$, and the indicators are equal on $\{\Delta_t = 0\}$

Lower bound - I. Sum

Write $I_j = \{0, \ldots, j\}$, $j \in \mathbb{Z}_+$. Then

$$\begin{aligned} d_t(x, x') &\geq P_x(X_t \in I_j) - P_{x'}(X_t \in I_j) \\ &= \sum_{k=x}^{x'-1} P_k(X_t \in I_j) - P_{k+1}(X_t \in I_j) \\ &= \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X'_t)] \\ &= \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X'_t), \Delta_t = 1] \end{aligned}$$

Continued on next slide...

Lower bound - I. Sum, continued

We showed

$$d_t(x,x') \geq \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X_t'), \Delta_t = 1]$$

Lower bound - I. Sum, continued

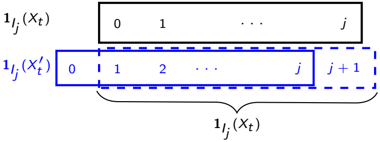
We showed

$$d_t(x,x') \geq \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X_t'), \Delta_t = 1]$$

$$= \sum_{k=x}^{x'-1} -P_{k,k+1}(X_t' = 0, \Delta_t = 1) + P_{k,k+1}(X_t' = j+1, \Delta_t = 1)$$

Explanation

On $\{\Delta_t = 1\}$, black - solid blue = dashed blue - solid blue



Lower bound - I. Sum, continued

We showed

$$d_t(x,x') \geq \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X_t'), \Delta_t = 1]$$

$$= \sum_{k=x}^{x'-1} \underbrace{-P_{k,k+1}(X_t' = 0, \Delta_t = 1)}_{(*)} + \underbrace{P_{k,k+1}(X_t' = j+1, \Delta_t = 1)}_{(**)}$$

$$= 0 + \sum_{k=x}^{x'-1} \underbrace{P_{k,k+1}(X_t = j, \Delta_t = 1)}_{(**)}$$

Explanation

On $\{\Delta_t = 1\}$, $X_t' = X_t + 1 > 0$.

Lower bound - I. Sum, continued

We showed

$$\begin{aligned} d_t(x,x') &\geq \sum_{k=x}^{x'-1} E_{k,k+1}[\mathbf{1}_{I_j}(X_t) - \mathbf{1}_{I_j}(X'_t), \Delta_t = 1] \\ &= \sum_{k=x}^{x'-1} -P_{k,k+1}(X'_t = 0, \Delta_t = 1) + P_{k,k+1}(X'_t = j+1, \Delta_t = 1) \\ &= 0 + \sum_{k=x}^{x'-1} P_{k,k+1}(X_t = j, \Delta_t = 1) \end{aligned}$$

Lower bound - I. Sum, continued

Lower bound - I. Sum, conclusion

Lemma 2

Suppose $0 \le x < x'$*.*

$$d_t(x, x') \ge \max_{j \in \mathbb{Z}_+} \sum_{k=x}^{x'-1} P_{k,k+1}(X_t = j, \Delta_t = 1). \tag{2}$$

Note:

- Coupling (normally used for upper bound) is part of statement through Δ_t.
- Argument works for any MC on $\mathbb{Z}_+$ and coupling with $1 = \Delta_0 \ge \Delta_1 \ge \dots$.

Lower Bound - II. Parameter change, reminder

- Last lemma

$$d_t(x, x') \geq \max_{j \in \mathbb{Z}_+} \sum_{k=x}^{x'-1} P_{k,k+1}(X_t = j, \Delta_t = 1)$$

Lower Bound - II. Parameter change, reminder

- Last lemma
- Will show parameter change

$$d_t(x,x') \geq \max_{j\in\mathbb{Z}_+} \sum_{k=x}^{x'-1} \boxed{P_{k,k+1}(X_t = j, \Delta_t = 1)}$$

$$\Vert$$

$$\boxed{\alpha^t P_k^{\tilde{p}}(X_t = j)}$$

Lower Bound - II. Parameter change, reminder

- Last lemma
- Will show parameter change
- $\Rightarrow$ proof of Theorem 1 is $\square$

$$d_t(x, x') \geq \max_{j \in \mathbb{Z}_+} \sum_{k=x}^{x'-1} P_{k,k+1}(X_t = j, \Delta_t = 1)$$

$$\|$$

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Time to derive...

$$d_t(x, x') \geq \max_{j \in \mathbb{Z}_+} \sum_{k=x}^{x'-1} P_{k,k+1}(X_t = j, \Delta_t = 1)$$

$$\|$$

$$\alpha^t P_k^{\tilde{p}}(X_t = j)$$

Lower bound - II. Parameter change

- Condition on N_t, number of catastrohes up to time t:

$$
\begin{aligned}
P_{k,k+1}(X_t = j, \Delta_t = 1 | N_t = n) &= P_{k,k+1}(X_t = j | N_t = n) P_{k,k+1}(\Delta_t = 1 | N_t = n) \\
&= P_{k,k+1}(X_t = j | N_t = n)(1-c)^n \qquad (3)
\end{aligned}
$$

  Because, conditioned on N_t
  - X_t and Δ_t are independent, and
  - $(\Delta_t | N_t = n) \sim \text{Bern}(1-c)^n$.

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- Multiply by $P(N_t = n)$:

$$P_{k,k+1}(X_t = j, \Delta_t = 1, N_t = n) \stackrel{(3)}{=} P_{k,k+1}(X_t = j | N_t = n)(1-c)^n P(N_t = n) \tag{4}$$

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- Change parameter:

$$(1-c)^n P(N_t = n) = \alpha^t P(\mathrm{Bin}(t, \tilde{p}) = n) = \alpha^t P^{\tilde{p}}(N_t = n). \tag{5}$$

Because change of measure formula from binomial with success parameter p to $\tilde{p}$

Lower bound - II. Parameter change

- Condition on N_t, number of catastrohes up to time t:

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- Putting it all together

$$P_{k,k+1}(X_t = j, \Delta_t = 1) = \sum_{n \in \mathbb{Z}_+} P_{k,k+1}(X_t = j, \Delta_t = 1, N_t = n) \overset{(4)(5)}{=} \sum_{n \in \mathbb{Z}_+} P_k(X_t = j | N_t = n) \alpha^t P^{\tilde{p}}(N_t = n) = \alpha^t P_k^{\tilde{p}}(X_t = j).$$

Because the distribution of $(X_t | N_t)$ is independent of the parameter p.

Lower bound - II. Parameter change

- Condition on N_t, number of catastrohes up to time t:

$$P_{k,k+1}(X_t = j, \Delta_t = 1 | N_t = n) = P_{k,k+1}(X_t = j | N_t = n) P_{k,k+1}(\Delta_t = 1 | N_t = n)$$
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- Putting it all together

$$P_{k,k+1}(X_t = j, \Delta_t = 1) = \alpha^t P_k^{\tilde{p}}(X_t = j).$$

□

Poisson Limit

Assumption

$$\begin{cases} p_n \to 0 \\ \frac{p_n}{c_n} \to \beta \in (0, \infty) \end{cases} \qquad (\star)$$

In the sequel, we write $P^{(n)}_{\cdot}, \pi^{(n)}, \ d^{(n)}_{\cdot}(\cdot, \cdot)$ for the respective quantities.

Limit Process

Theorem 3

Assume $(\star)$*. Then the family of rescaled processes* $Y^{(n)}_s = X_{\lfloor s/c_n \rfloor}, \ s \in \mathbb{R}_+$*, converges in distribution to a continuous-time Markov chain on* $\mathbb{Z}_+$ *with rates:*

$$\lambda(x, y) = \begin{cases} \beta & y = x + 1 \\ x & x > 0, \ y = x - 1 \\ 0 & \textit{otherwise} \end{cases}$$

Corollary 3

Under $(\star)$*,*

$$\pi^{(n)} \to \mathit{Pois}(\beta),$$

the stationary distribution of the limit chain.

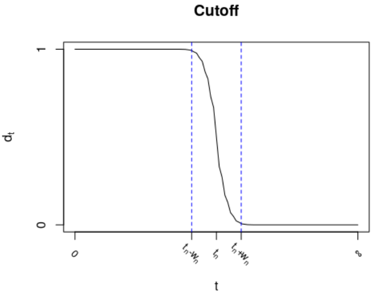
Cutoff

What is cutoff?

We say that the family of TFs and initial distributions μ_n exhibits a cutoff at t_n with window w_n if there exists a sequence $t_n \to \infty$ and $w_n = o(t_n)$ such that for $\alpha > 0$,

- $d^{(n)}_{t_n - \alpha w_n}(\mu_n, \pi^{(n)}) \to 1.$
- $d^{(n)}_{t_n + \alpha w_n}(\mu_n, \pi^{(n)}) \to 0.$

A sharp transition from being "orthogonal" to stationary distribution to being stationary.



Examples for Cutoff

Usually families of finite-state reversible chains.

- Lazy RW on the n-dimensional hypercube.
- RWs on $\{0, \ldots, n\}$ with constant drift to the right.

More? Slides by David Levin https://pages.uoregon.edu/dlevin/TALKS/durham.pdf

Our cutoff results

Recall ($\star$): $p_n \to 0$ and $p_n/c_n \to \beta$, so $\pi^{(n)} \to \text{Pois}(\beta)$.

Theorem 4

Suppose that $y_n \to \infty$. *Let* $t_n = \frac{\ln y_n}{c_n}$. *Then for every* $\epsilon > 0$,

1. $\lim_{n\to\infty} \inf_{t<t_n-b_n} d_t^{(n)}(y_n, \pi^{(n)}) = 1$, *where*

$$b_n = (1+\epsilon)\left(\frac{1}{2}\ln y_n + \frac{\ln\ln y_n}{c_n}\right).$$

2. $\lim_{\epsilon\to 0+} \limsup_{n\to\infty} \sup_{t>t_n+\frac{1}{\epsilon c_n}} d_t^{(n)}(y_n, \pi^{(n)}) = 0.$

In other words, a cutoff at time $t_n = \ln y_n / c_n$ with window $O(\max(\ln y_n, \frac{\ln\ln y_n}{c_n}))$.

Why $y_n \to \infty$?

Otherwise, $d_0(y_n, \pi^{(n)}) = \|\delta_{y_n} - \pi^{(n)}\|_{TV}$ is uniformly < 1, so part 1 cannot hold true.

Fim. Obrigado!